Shannon information, LMC complexity and Rényi entropies: a straightforward approach

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Abstract

The LMC complexity, an indicator of complexity based on a probabilistic description, is revisited. A straightforward approach allows us to establish the time evolution of this indicator in a near-equilibrium situation and gives us a new insight for interpreting the LMC complexity for a general non equilibrium system. Its relationship with the Rényi entropies is also explained. One of the advantages of this indicator is that its calculation does not require a considerable computational effort in many cases of physical and biological interest.
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1. Shannon information

Entropy plays a crucial theoretical role in physics of macroscopic equilibrium systems. The probability distribution of accessible states of a constrained system in equilibrium can be found by the inference principle of maximum entropy [1]. The macroscopic magnitudes and the laws that relate them can be calculated with this probability distribution by standard statistical mechanics techniques.

The same scheme could be thought for extended systems far from equilibrium, but in this case we do have neither a method to find the probability distribution nor the knowledge of the relevant magnitudes bringing the information that can predict the system's behavior. It is not the case, for instance, with the metric properties of low dimensional chaotic systems by means of the Lyapunov exponents, invariant measures and fractal dimensions [2].

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Shannon information or entropy H [3] can still be used as a magnitude in a general situation with N accessible states:

$$H = K \sum_{i=1}^{N} p_i \log p_i \qquad (1)$$

with K a positive real constant and p_i the normalized associated probabilities, $\sum_{i=1}^{N} p_i = 1$. An isolated system in equilibrium presents equiprobability, $p_i = 1/N$ for all i, among its accessible states and this is the situation of maximal entropy,

$$H_{\max} = K \log N. \qquad (2)$$

If the system is out of equilibrium, the entropy H can be expanded around this maximum $H_{\max}$:

$$H(p_1, p_2, \ldots, p_N) = K \log N - \frac{NK}{2} \sum_{i=1}^{N} \left(p_i - \frac{1}{N} \right)^2 + \ldots$$

$$= H_{\max} - \frac{NK}{2} D + \ldots \qquad (3)$$

where the quantity $D = \sum_i (p_i - \frac{1}{N})^2$, that we call disequilibrium, is a kind of distance from the actual system

configuration to the equilibrium. If the expression (Eq. (3)) is multiplied by H we obtain:

$$H^2 = H \cdot H_{\max} - \frac{NK}{2} H \cdot D + K^2 f(N, p_i), \qquad (4)$$

where $f(N, p_i)$ is the entropy multiplied by the rest of the Taylor expansion terms, which present the form $\frac{1}{N} \sum_i (Np_i - 1)^m$ with $m > 2$. If we rename $C = H \cdot D$,

$$C = \text{cte} \cdot H \cdot (H_{\max} - H) + K\bar{f}(N, p_i), \qquad (5)$$

with $\text{cte}^{-1} = NK/2$ and $\bar{f} = 2f/N$. The idea of distance for the disequilibrium is now clearer if we see that D is just the real distance $D \sim (H_{\max} - H)$ for systems in the vicinity of the equiprobability. In an ideal gas we have $H \sim H_{\max}$ and $D \sim 0$, then $C \sim 0$. Contrarily, in a crystal $H \sim 0$ and $D \sim 1$, but also $C \sim 0$. These two systems are considered as classical examples of simple models and are extrema in a scale of disorder (H) or disequilibrium (D) but those should present null complexity in a hypothetic measure of complexity. This last asymptotic behavior is verified by the variable C (Fig. 1) and C has been proposed as a such magnitude [4]. We formalize this simple idea recalling the recent definition of LMC complexity in the next section.

Let us see another important property arising from relation (5). If we take the time derivative of C in a neighborhood of equilibrium by approaching $C \sim H(H_{\max} - H)$, then we have

$$\frac{dC}{dt} \sim - H_{\max} \frac{dH}{dt} \cdot \qquad (6)$$

The irreversibility property of H implies that $dH/dt \geq 0$, the equality occurring only for the equipartition, therefore

$$\frac{dC}{dt} \leq 0. \qquad (7)$$

Hence, in the vicinity of $H_{\max}$, LMC complexity is always decreasing on the evolution path towards equili-
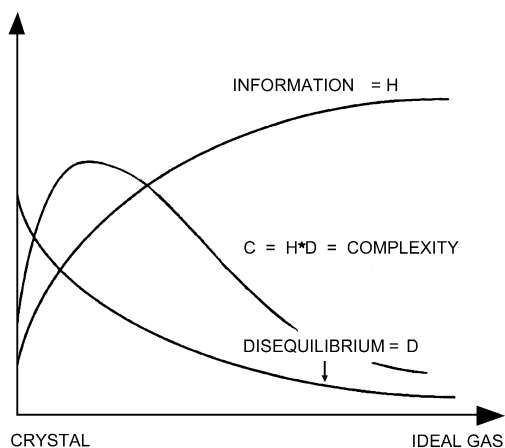

Fig. 1. Sketch of physical intuition of the magnitudes "information" (H) and "disequilibrium" (D) between the two simple systems: crystal and ideal gas. Also, intuitive behavior required for the magnitude complexity. The quantity $C = H \cdot D$ was proposed as such a magnitude.

brium, independently of the kind of transition and of the system under study. This does not forbid that complexity can increase when the system is very far from equilibrium. In fact this is the case in a general situation as it can be seen, for instance, in the system presented in [5].

2. LMC complexity

Let us assume that at the scale of observation a system has N accessible states $\{x_1, x_2, \ldots, x_N\}$ (N-system) and a probability distribution $\{p_1, p_2, \ldots, p_N\}$ of each state ($p_i \neq 0$ for all i). Then, at this level of description the knowledge of the underlying physical laws is expressed by a probability distribution among the accessible states. Shannon and Weaver [3] demonstrated that the only function that gives the information of a system under the most elementary assumptions is $H = - K \sum_{i=1}^{N} p_i \log p_i$. It is easy to find out that the information H contained in a crystal is $H_c \sim 0$, while for an isolated gas $p_i \sim 1/N$ and then $H_g \sim K \log N$ that represents the maximum of information for a system with N states. Any other N-system will have information between those two extrema.

The disequilibrium D of a system can be taken as some kind of distance to an equiprobable distribution. Two conditions are required to this magnitude $D > 0$ (positive measure of complexity) and $D = 0$ in the limit of equiprobability. The easier solution is to add the quadratic distances of each state to the equiprobability, i.e., $D = \sum_{i=1}^{N} \left(p_i - \frac{1}{N} \right)^2$. This function will be a maximum for a crystal and zero (by construction) for an ideal gas. Any other N-system will have disequilibrium between these two extrema.

Following the discussion in the introduction the definition of LMC complexity (C) is presented [4]:

$$C = H \cdot D = - \left(K \sum_{i=1}^{N} p_i \log p_i \right) \cdot \left(\sum_{i=1}^{N} \left(p_i - \frac{1}{N} \right)^2 \right) \qquad (8)$$

This definition fits the intuitive arguments and gives $C \sim 0$ for the two systems (a perfect crystal and the ideal gas) discussed herein. Any other system will have an intermediate behavior and therefore $C > 0$. At different scales a different number of states (a different probability distribution) are accessible to the system and therefore different H and D. Therefore, the magnitude complexity defined is scale-dependent as expected.

Direct simulations of the definition of C and its comparison with H for several systems in different contexts have been presented in [4,5]. For a 2-system an analytical expression for the curve $C(H)$ is obtained. For $N > 2$ the relationship between H and C is not univoque anymore. Many different distributions $\{p_i\}$ can have associated the same information H but different complexity C. But a similar form as the one found in the 2-system case is

recovered for an N-system when the maximum complexity $C_{\max}(H)$ is calculated for each H. These curves, represented in a normalized way $\bar{C}_{\max}(\bar{H})$, with normalization K=1/log N, have been numerically computed for the cases N=2,3,5,7 in [4]. When N tends towards infinity [5] the maximum disequilibrium scales as $(1-\bar{H})^2$ and the maximum complexity tends to

$$\lim_{N\to\infty} \bar{C}_{\max} = \bar{H}\cdot\left(1-\bar{H}\right)^2\cdot \tag{9}$$

The limit of the minimum disequilibrium and complexity vanishes,

$$\lim_{N\to\infty} \bar{C}_{\min} = 0. \tag{10}$$

In general, in the limit $N\to\infty$, the complexity is not a trivial function of the entropy, in the sense that for a given H there exists a range of complexities between 0 and $C_{\max}(H)$ (Eqs. (9), (10)). In particular, in this asymptotic limit, the maximum of $\bar{C}_{\max}$ is found when $\bar{H}$=1/3, which gives a maximum of the maximum complexity of $\bar{C}_{\max}$=4/27 and confirms the numerical calculation of [6]. This value is reached when the distribution presents a dominant state with probability $p_{\max}$=2/3 and the rest of the infinitely many states is a uniform 'sea' of equal probability. We say that this distribution is a type-like 'Rey–Pueblo' configuration.

An attempt of extending the LMC complexity for continuous systems has been performed in [7]. When the number of states available for a system is a continuum then the natural representation is a continuous distribution. In this case, the entropy can become negative. The positivity of C for every distribution is recovered by taking the exponential of H. If we define $\hat{C}=\hat{H}\cdot D=e^H\cdot D$ as an extension of C to the continuous case interesting properties characterizing the indicator $\hat{C}$ appear. Namely, its invariance under translations, rescaling transformations and replication convert $\hat{C}$ in a good candidate to be considered as an indicator bringing essential information about the statistical properties of a continuous system.

The most important point is that the definition should work in systems out of equilibrium. We have a significative example [4] in the logistic map, the typical chaotic system in which the transition from chaos to a 3-period orbit via intermittency is known to be extremely "complex". This is due to the fact that the intermittent bursts are more and more improbable and impredictible when the transition point is approached. After the transition point a period three orbit stabilizes and the dynamics becomes simple. (Complexity is calculated by means of the binary sequences issued from the numerical simulations of the map, using the natural partition). We see in Fig. 3a of [4] that the values of C in the intermittency transition point recalls a second-order phase transition.

3. Rényi entropies

Generalized entropies have been introduced by Rényi [8] in the form of

$$I_q = \frac{1}{1-q}\log\left(\sum_{i=1}^{N} p_i^q\right), \tag{11}$$

where q is an index running over all the integer values. By differentiating I_q with respect to q a negative quantity is obtained independently of q, then I_q monotonously decreases when q increases.

The Rényi entropies are an extension of the Shannon information H. In fact, H is obtained in the limit $q\to1$:

$$H = I_1 = \lim_{q\to1} I_q = -\sum_{i=1}^{N} p_i\log p_i, \tag{12}$$

where the constant K of Eq. (1) is considered to be the unity. The disequilibrium D is also related with $I_2 = -\log(\sum_{i=1}^{N} p_i^2)$. We have that

$$D = \sum_{i=1}^{N} p_i^2 - \frac{1}{N} = e^{-I_2} - \frac{1}{N}, \tag{13}$$

then the LMC complexity is

$$C = H\cdot D = I_1\cdot\left(e^{-I_2} - \frac{1}{N}\right). \tag{14}$$

The behavior of C in the neighborhood of $H_{\max}$ takes the form

$$C \sim \frac{1}{N}\left(\log^2 N - I_1 I_2\right), \tag{15}$$

The obvious generalization of the Rényi entropies for a normalized continuous distribution $p(x)$ is

$$I_q = \frac{1}{1-q}\log\int [p(x)]^q \mathrm{d}x. \tag{16}$$

Hence,

$$H = I_1 = -\int p(x)\log p(x)\mathrm{d}x, \tag{17}$$

$$D = e^{-I_2} = \int [p(x)]^2 \mathrm{d}x. \tag{18}$$

The dependence of $\hat{C}=e^H\cdot D$ with I_1 and I_2 yields

$$\log\hat{C} = (I_1 - I_2). \tag{19}$$

This indicates that a family of different indicators could derive from the differences established among Rényi entropies with different q-indices. Let us remark at this point the coincidence of the indicator log $\hat{C}$ with the quantity S_{str} introduced by Varga and Pipek as a meaningful parameter to characterize the shape of a distribution. They apply this formalism to the Husimi representation, i.e., to

the projection of wave functions onto the coherent state basis [9].

The invariance of $\hat{C}$ under rescaling transformations implies that this magnitude is conserved in many different processes. For instance, the initial Gaussian-like distribution will continue to be Gaussian in a classical diffusion process. Then $\hat{C}$ is constant in time: $d\hat{C}/dt=0$, and we have:

$$\frac{dI_1}{dt} = \frac{dI_2}{dt}.$$ (20)

The equal losing rate of I_1 and I_2, i.e., the synchronization of both quantities, is the cost to be paid in order to maintain the shape of the distribution associated to the system and, hence, all its statistical properties will remain unchanged during its time evolution.

4. Discussion and conclusions

A definition of complexity (LMC complexity) based on a probabilistic description of physical systems has been revisited. This definition contains basically an interplay between the information contained in the system and the distance to equipartition of the probability distribution representing the system. Besides giving the main features of an intuitive notion of complexity, we showed that it allows to successfully discern situations considered as complex in systems of a very general interest. In particular it proved to be useful for quantifying the complexity in some statistical and laser systems [10,11]. Its relationship with the Shannon information and the generalized Rényi entropies has been shown to be explicit. Moreover it has been possible to establish the decrease of this magnitude when a general system evolves from a near-equilibrium situation to the equipartition.

Many different notions of complexity have been proposed until now, mainly in the context of computational and social sciences. Most of these definitions present either operational difficulties or arise logical problems. The main advantage of LMC complexity is its generality and the fact that it is operationally simple and do not require a big amount of calculations [12]. This advantage has been worked out in different examples, such as the study of the time evolution of C for a simplified model of an isolated gas, the "tetrahedral gas" [5], the slight modification of C as an effective method by which the complexity in hydrological systems can be identified [13], the attempt of generalize C in a family of simple complexity measures [14,15], several expressions for quantum localization and quantum entropy [16], some statistical features of the behavior of C for DNA sequences [17], some wavelet-based informational tools used to analyze the brain electrical activity in epilectic episodes in the plane of coordinates (H,C) [18] and as a method to discern complexity in two-dimensional patterns [19]. We are convinced that it provides a useful way of thinking and it can help in the future to gain more insight on the physical grounds of models with potential biological interest.

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